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LOLLIPOP GRAPHS AND THEIR PARTITIONS BASED ON LAPLACIAN MATRICES

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ABSTRACT

The Lollipop graph $LP_{n,m}$ defined in this paper is the coalescence of a complete graph K_n and a path P_m with a pendent vertex. Lollipop graph defined as a coalescence of a cycle and a path is already studied from the view point of their spectrum and their cospectral properties [1]. However, among various connected, finite graphs, we are interested in partitioning techniques. Spectral clustering methods use eigenvalues and eigenvectors of associated matrices of graphs, where Laplacian matrices play a vital role in finding clusters. There are some other non-deterministic polynomial-time hard techniques to find clusters in a graph. Minimum normalized cut is the one, which is widely used in image segmentation. But there are some differences between the partitions generated by these techniques. We are interested in finding graphs, which perform poorly on spectral clustering methods. The lollipop graph is one of the counter example graphs. In this research, we find the general formula for the characteristic polynomial of difference Laplacian matrix of a lollipop graph $LP_{n,m}$. We find sign graphs based on their eigenvectors corressponding to the second smallest eigenvalue. We reviewed the formula for the minimum normalized cut of a lollipop graph $LP_{n,m}$ [2]. Finally, we compare the partitions of lollipop graph

generated by the spectral clustering method and the minimum normalized cut.

Keywords: Lollipop graph, Laplacian matrix, characteristic polynomials, eigenvalue, eigenvectors, minimum normalized cut.

INTRODUCTION

In algebraic graph theory, matrix theory and linear algebra are used to analyze adjacency matrix, difference Laplacian matrix and normalized Laplacian matrix. Algebraic methods are especially effective in treating graphs which are real symmetric. Spectral graph theory is the study and exploration of graphs through the eigenvalues and eigenvectors of matrices naturally associated with those graphs. Interesting properties of graphs should be revealed by these eigenvalues and eigenvectors.

There are several types of partitioning methods. Clustering or partitioning method, that use eigenvalues and eigenvectors of matrices associated with graph is called spectral clustering method. Partitioning of graphs can be done by using minimum normalized cut introduced by Shi and Malik. Eigenvectors of difference Laplacian matrix, normalized Laplacian matrix or adjacency matrix with negative off diagonal entries can be used to identify the number of connected sign graphs of a given graph based on their eigenvalues and eigenvectors. The goal of the partitioning is to find groups such that entities within same groups are similar and entities with different groups are dissimilar.

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complexity of graph with longer path, theeigenvalues and eigenvectors popilloL a fo graph $LP_{n,2}$ are considered to analyze their partitions. Spectral clustering methods use eigenvalues and eigenvectors of associated matrices of graphs, where Laplacian matrices play a vital role in finding clusters. Clustering methods, used in this research are based on sign patterns of eigenvectors corresponding to the second smallest eigenvalue and minimum normalized cut method of a Lollipop graph $LP_{n,2}$.

PRELIMINARIES

A graph is an ordered pair $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is a finite set called vertex set and $E(G)$ consists of two element subsets of $V(G)$ called edge set. Two vertices v_i and v_j of G are called adjacent if $\{v_i, v_j\} \in E(G)$. The order of G is the number of vertices in G. The size of G is the number of edges.

Definition 1:Difference Laplacian Matrix

The difference Laplacian matrix of G is the $n \times n$ matrix $L = (l_{ij})_{n \times n}$ defined as

$$l_{ij} = \begin{cases} d_i & if\ v_i = v_j, \\ -1 & if\ (v_i, v_j) \in E\ and\ v_i \neq v_j, \\ 0 & otherwise. \end{cases}$$

Definition 2(Weak Sign Graph)

A weak positive (negative) sign graph S is a maximal, connected subgraph of $G = (V, E)$, on vertices $v_i \in V$ with $u_i \geq 0$ $(u_i \leq 0)$ and with at least one $v_i \in V$ having $u_i > 0$ $(u_i < 0)$.

Definition 3(Strong Sign Graph)

A strong positive (negative) sign graph S is a maximal, connected subgraph of G, on vertices $v_i \in V$ with $u_i > 0$ $(u_i < 0)$.

Definition 4(Graph cut)

A subset of edges which disconnects the graph is called a **graph cut**. Let $G = (V, E, w)$ be a weighted graph and $W = (w_{ij})$ the weighted adjacency matrix. Then for $A, B \subset V$ and $A \cap B = \Phi$, the graph cut is denoted by

$$cut\ (A, B) = \sum_{i \in A, j \in B} w_{ij}$$

Definition 5(Normalized cut)

Let $G = (V, E)$ be a connected graph. $Let\ A, B \subset V, A \neq \Phi B \neq \Phi$ and $A \cap B = \Phi$The normalized cut, $Ncut(A, B)$ of G is defined by

$$Ncut(A, B) = cut\ (A, B)\left(\frac{1}{vol(A)} + \frac{1}{vol(B)}\right)$$

Definition 6(Minimum normalized cut ($Mcut(G)$)

Let $G = (V, E)$ be a connected graph. The minimum normalized cut, $Mcut(G)$ is defined by

$Mcut(G) = min\{Mcut_j(G) \mid j = 1,2,..\ .\}$,

where

$Mcut_j\ (G) = min\ \{\ Ncut(A, V \backslash A) \mid A \subset V,\ cut(A, V \backslash A) = j,$ $A\ and\ V \backslash A\ are\ connected.\}$

Definition 7:Lollipop Graph

A lollipop graph $LP_{n,m}\,(n \geq 3, m \geq 1)$ is obtained by connecting a vertex of complete Graph, $K_n = (V_K, E_K)$ to the end vertex of path, $P_m = (V_P, E_P)$, where $V_P = \{x_1, x_2, \ldots, x_m\}$ and$V_K = \{\ y_1, y_2, \ldots, y_n\ \}$. Define $LP_{n,m} = (V, E)$ as follows:

$$V = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$$

$$E = \{\ (x_i, x_{i+1}) | 1 \leq i \leq m-1\} \cup \{\ (y_i, y_j) \mid i \neq j, 1 \leq i \leq n, 1 \leq j \leq n\ \} \cup \{\ (x_m, y_1)\}.$$

Let $E_P = \{(x_i, x_{i+1}) | 1 \leq i \leq m-1\}$ and $E_K = \{\ (y_i, y_j) \mid i \neq j, 1 \leq i \leq n, 1 \leq j \leq n\ \}$. Then,

$$E = E_P \cup E_K \cup \{\ (x_m, y_1)\}.$$

Definition 8

For $v \in V(G)$, let $L_v(G)$ be the principal sub matrix of $L(G)$ formed by deleting the row and column corresponding to vertex v.

Lemma 1[3]

Let uv be a cut edge of a graph G. Let $G - uv = G_1 + G_2$, where G_1 and G_2 are the components of $G - uv$, $G_1 + G_2$ is the sum of G_1 and G_2, $u \in V(G_1)$ and $v \in V(G_2)$.

Then,

$$\Phi(L(G)) = \Phi(L(G_1))\,\Phi(L(G_2)) - \Phi(L(G_1))\,\Phi(L_v(G_2)) - \Phi(L_u(G_1))\Phi(L(G_2))$$

Here $\Phi(L(G)) = \det(\lambda I - L(G))$ and Φ be a Laplacian characteristic polynomial.

Lemma 2[2]

Let $n \geq 3$ and $m \geq 1$.We have the followings for a lollipop graph $LP_{n,m}$.

1. $c(\alpha - 1) < c(\alpha)\ iff\ m > \frac{1}{2}(n^2 - n + 4)(2 \leq \alpha \leq m$.
2. $c(m) \leq Ncut(A_2(\beta), V \backslash A_2(\beta))(1 \leq \beta < n)$.
3. If $m \leq \frac{1}{2}(n^2 - n + 4)$, then $c(m) \leq Ncut(B(\alpha,\beta), V\backslash B\alpha,\beta,$

$$(1 \leq \alpha \leq m - 2, 1 \leq \beta < n)$$

RESULTS

Laplacian characteristic polynomial of a Lollipop graph $LP_{n,m}$

According to the equation of the Laplacian characteristic polynomial of any graph, we have the Laplacian characteristic polynomial of a Lollipop graph as

$$\Phi\left(L(LP_{n,m})\right) = \Phi\left(L(LP_{n,1})\right)\Phi(L(P_{m-1})) - \Phi\left(L(LP_{n,1})\right)\Phi(L_v(P_{m-1}))$$

$$-\Phi\left(\mathrm{L}_u\left(LP_{n,1}\right)\right)\Phi\left(\mathrm{L}(P_{m-1})\right)$$

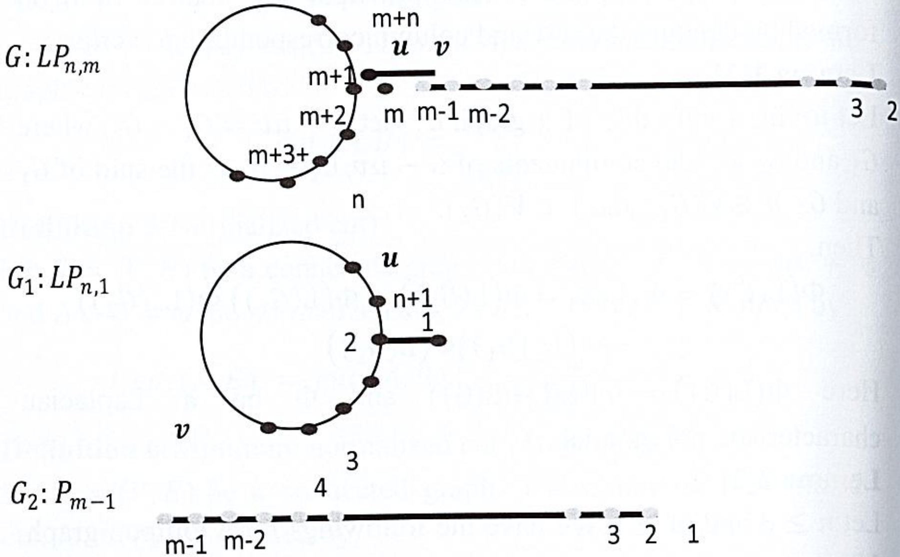


Figure 1

The Laplacian matrix of a graph $LP_{n,1}$ is

$$L\left(LP_{n,1}\right) = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots\cdots & 0 & 0 \\ -1 & n & -1 & -1 & \cdots\cdots & -1 & -1 \\ 0 & -1 & (n-1) & -1 & \cdots\cdots & -1 & -1 \\ 0 & -1 & -1 & (n-1) & \cdots\cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & -1 & -1 & \cdots\cdots & (n-1) & -1 \\ 0 & -1 & -1 & -1 & \cdots\cdots & -1 & (n-1) \end{pmatrix}_{(n+1)(n+1)}$$

The characteristic polynomial of a $LP_{n,1}$ is, $\left|L\left(LP_{n,1}\right) - \lambda I\right| = 0$

$$\left|L\left(LP_{n,1}\right) - \lambda I\right| =$$

$$\begin{vmatrix} (1-\lambda) & -1 & 0 & 0 & \dots\dots & 0 & 0 \\ -1 & (n-\lambda) & -1 & -1 & \dots\dots & -1 & -1 \\ 0 & -1 & (n-\lambda-1) & -1 & \dots\dots & -1 & -1 \\ 0 & 1 & -1 & (n-\lambda-1) & \dots\dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & -1 & -1 & \dots\dots & (n-\lambda-1) & -1 \\ 0 & -1 & -1 & -1 & \dots\dots & -1 & (n-\lambda-1) \end{vmatrix}$$

$= (1-\lambda)\{-[2(\lambda - n) - 1]\,(-1)^{n-1}\,(\lambda - 1)\,(\lambda - n)^{n-2} - (\lambda - n)^2$ $[-(-1)^{n-3}\;(\lambda\;-\;2)\;(\lambda - n)^{n-3}]\;\}\;-\;(-1)^2(-1)^{n-1}(\lambda -$ $1\lambda{-}nn{-}2$

$$= (-1)^{n-1}(\lambda - n)^{n-2}\{(\lambda - 1)[2(\lambda - n) - 1](\lambda - 1)$$
$$+(-1)^2(1 - \lambda)(\lambda - 2)(\lambda - n) - (-1)^2(\lambda - 1)\}$$
$$= (-1)^{n-1}\lambda(\lambda - 1)(\lambda - (n+1))(\lambda - n)^{n-2}$$

So, the characteristic polynomial of a graph $LP_{n,1}$ is,

$$\Phi\left(L\left(LP_{n,1}\right)\right) \Rightarrow \lambda(\lambda - 1)\left(\lambda - (n+1)\right)(\lambda - n)^{n-2} = 0$$

(1)

The difference Lalacian matrix of a path P_{m-1} is

$$L(P_{m-1}) = \begin{pmatrix} 1 & -1 & 0 \dots & 0 & 0 \\ -1 & 2 & -1 \dots & 0 & 0 \\ 0 & -1 & 2 \dots & 0 & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & 0 \dots & 2 & -1 \\ 0 & 0 & 0 \dots & -1 & 2 \end{pmatrix}_{(m-1)(m-1)}$$

We know that the Lalacian characteristic polynomial of a path P_{m-1} is

$$\Phi\big(\mathrm{L}(P_{m-1})\big) \Rightarrow \left[\lambda - \left(2 - 2cos\left(\frac{\pi j}{m-1}\right)\right)\right] = 0$$

(2)

where $j = 0, 1, 2, \dots, m-2$

Similarly we can write the Lalacian characteristic polynomial of a $\mathrm{L}_v(P_{m-1})$ is

$$\Phi\big(\mathrm{L}_v(P_{m-1})\big) \equiv \Phi\big(\mathrm{L}(P_{m-2})\big) \Rightarrow \left[\lambda - \left(2 - 2cos\left(\frac{\pi i}{m-2}\right)\right)\right] = 0$$

(3)

where $i = 0, 1, 2, \dots, m-3$

After deleting the row and column corresponding to vertex v from the Laplacian matrix $L(LP_{n,1})$, we get

$$\mathrm{L}_u\big(LP_{n,1}\big) = \begin{pmatrix} n & -1 & -1 & \dots & -1 & -1 \\ -1 & (n-1) & -1 & \dots & -1 & -1 \\ -1 & -1 & (n-1)\dots & & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1\dots & & (n-1) & -1 \\ -1 & -1 & -1\dots & & -1 & (n-1) \end{pmatrix}_{n\times n}$$

Then,

$$\Phi\left(\mathrm{L}_u\big(LP_{n,1}\big)\right) \Rightarrow \left|\mathrm{L}_u\big(LP_{n,1}\big) - \lambda I\right| = 0$$

$$(\lambda - n)^{n-2}(\lambda^2 - (n+1)\lambda + 1) = 0 \qquad (4)$$

According to the equations (1), (2), (3) and (4) , we can get the general formula for the characteristic polynomial of difference Laplacian matrix of a Lollipop graph $LP_{n,m}$.

Then,

$$\Phi\left(\mathrm{L}\left(LP_{n,m}\right)\right) = \Phi\left(\mathrm{L}\left(LP_{n,1}\right)\right)\ \Phi\left(\mathrm{L}(P_{m-1})\right) - \Phi\left(\mathrm{L}\left(LP_{n,1}\right)\right)\ \Phi\left(\mathrm{L}_v(P_{m-1})\right)$$

$$-\Phi\left(\mathrm{L}_u\left(LP_{n,1}\right)\right)\Phi\left(\mathrm{L}(P_{m-1})\right)=0$$

$$\lambda(\lambda-1)(\lambda-(n+1))(\lambda-n)^{n-2}\times\left[\lambda-\left(2-2cos\left(\frac{\pi j}{m-1}\right)\right)\right]$$
$$-\lambda(\lambda-1)(\lambda-(n+1))(\lambda-n)^{n-2}$$
$$\times\left[\lambda-\left(2-2cos\left(\frac{\pi i}{m-2}\right)\right)\right]$$
$$-(\lambda-n)^{n-2}(\lambda^2-(n+1)\lambda+1)$$
$$\times\left[\lambda-\left(2-2cos\left(\frac{\pi j}{m-1}\right)\right)\right]=0$$

Where$i = 0, 1, \dots, m-3$ and $j = 0, 1, \dots, m-2$

⟹

$$(\lambda-n)^{n-2}\{\left[\lambda-\left(2-2cos\left(\frac{\pi j}{m-1}\right)\right)\right][\lambda(\lambda-1)(\lambda-(n+1)) - (\lambda^2-(n+1)\lambda+1)]$$
$$-\lambda(\lambda-1)(\lambda-(n+1))\left[\lambda-\left(2-2cos\left(\frac{\pi i}{m-2}\right)\right)\right]\}=0$$

⟹

$$(\lambda-n)^{n-2}\{\left[\lambda-\left(2-2cos\left(\frac{\pi j}{m-1}\right)\right)\right][\lambda^3-(n+3)\lambda^2 + 2(n+1)\lambda-1]$$
$$-\lambda(\lambda-1)(\lambda-(n+1))\left[\lambda-\left(2-2cos\left(\frac{\pi i}{m-2}\right)\right)\right]\}=0$$

$i = 0, 1, \dots, m-3$ and $j = 0, 1, \dots, m-2$

So, this is the Laplacian characteristic polynomial of a Lollipop graph $LP_{n,m}$.

Butdue to the complexity, consider the $LP_{n,2}$ Lollipop graph to find the eigenvalue and eigenvectors of a Laplacian matrices. Then, The Laplacian matrix is,

$L(LP_{n,2}) =$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & n & -1 & \dots & -1 & -1 \\ 0 & 0 & -1 & n-1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & -1 & -1 & \dots & n-1 & -1 \\ 0 & 0 & -1 & -1 & \dots & -1 & n-1 \end{pmatrix}_{(n+2)\times(n+2)}$$

The characteristic polynomial is $|L - \lambda I| = 0$.

$|L - \lambda I| =$

$$\begin{vmatrix} 1-\lambda & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2-\lambda & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & n-\lambda & -1 & \dots & -1 & -1 \\ 0 & 0 & -1 & n-1-\lambda & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & -1 & -1 & \dots & n-1-\lambda & -1 \\ 0 & 0 & -1 & -1 & \dots & -1 & n-1-\lambda \end{vmatrix}_{(n+2)\times(n+2)}$$

$=(-1)^n(\lambda-n)^{n-2}\,[\,(\lambda^2 - 3\lambda + 1)\{\,[2\lambda^2 - (2n-1)\lambda - 2\lambda + (2n+1) - (\lambda^2 - 2\lambda - n\lambda + 2n)]\} - (\lambda-1)^2\,]$

$=(-1)^n\lambda\,(\lambda-n)^{n-2}\,[\,\lambda^3 - (n+4)\,\lambda^2 + (3n+4)\,\lambda - (n+2)]$

The characteristic polynomial of Lollipop graph $LP_{n,2}$ is,

$\lambda\,(\lambda-n)^{n-2}\,[\,\lambda^3 - (n+4)\,\lambda^2 + (3n+4)\,\lambda - (n+2)] = 0.$

Laplacian eigenvalues are $\lambda = 0$, $\lambda = n$ with multiplicity $(n-2)$ and the remaining eigenvalues are the solution of the equation

$\lambda^3 - (n+4)\,\lambda^2 + (3n+4)\,\lambda - (n+2) = 0$.

Let

$$f(\lambda) = \lambda^3 - (n+4)\,\lambda^2 + (3n+4)\,\lambda - (n+2)$$

$$f(0) = -(n+2) < 0$$

$$f(1) = 1^3 - (n+4)1^2 + (3n+4) - (n+2)$$

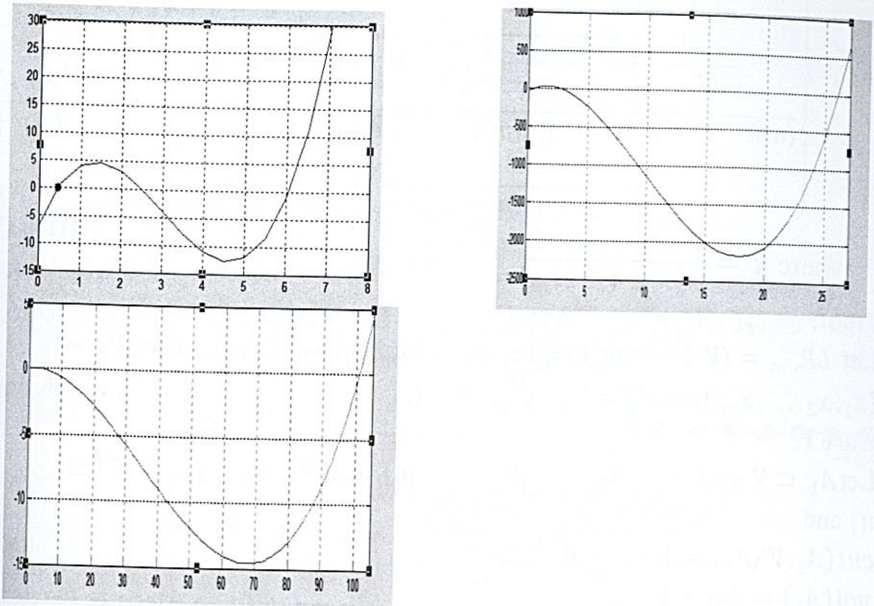
$$=n-1 \;>0 \qquad if n \geq 2$$

$$f(n) = n^3 - (n+4)n^2 + (3n+4)n - (n+2)$$

$$= -(n-1)(n-2) < 0 \qquad if\; n > 2$$

ehTremaining two eigenvalues are in the interval$0 < \lambda < 1$ and $1 < \lambda < n$.

Considering figure 2, it is also clear that the dnocessmallest root is less than 1.



(a).n = 5, m = 2 (b). n=25, m=2 (c) n=100, m=2

Figure 2: plot of $f(\lambda)$ vs λ of Laplacian matrix for n=5, n=25 and n=100

Observations:

Let λ_2 be the second smallest eigenvalue and $U = (u_1, \ldots, u_{n+2})$the second eigenvector of $LP_{n,2}$. Then $u_i = u_{i+1}$ for $i = 4, \ldots, n+2$.

We note that, when$u_1, u_2 > 0$ then $u_i < 0 \; for \; i = 3, \ldots, n+2$ and when$u_1, u_2 < 0$ then $u_i > 0$ for $i = 3, \ldots, n+2$.

Partition of a Lollipop graph

Minimum normalized cut of a Lollipop graph

Theorem 1[2]

Let $n \geq 3$ and $m \geq 1$. For a Lollipop graph $LP_{n,m}$,

$$Mcut(LP_{n,m}) = \begin{cases} \dfrac{2m+n(n-1)}{(2m-1)(1+n(n-1))}\left(2 \leq m \leq \dfrac{n^2-n+4}{2}\right), \\ \dfrac{4}{[2m+n(n-1)]}\left(k \in \mathbb{Z}\ and\ m > \dfrac{n^2-n+4}{2}\right), \\ \dfrac{4(2m+n(n-1))}{(n(n-1)+2(m-1))(2(m+1)+n(n-1))}\left(k \notin \mathbb{Z}\ and\ m > \dfrac{n^2-n+4}{2}\right), \\ \dfrac{2+n(n-1)}{(n+1)(n-1)}(m=1), \end{cases}$$

where $k = \dfrac{2m+n(n-1)+2}{4}$.

Proof:

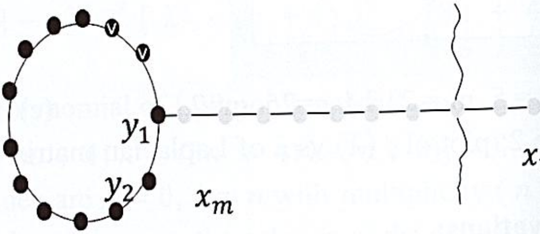
Let $LP_{n,m} = (V,E)$ with $V = \{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n\}$, where $V_P = \{x_1, x_2, \dots, x_m\}$ and $V_K = \{y_1, y_2, \dots, y_n\}$.

Case I:

Let $A_1 \subset V$ such that $A_1 = \{x_i \mid 1 \leq i \leq m\}$ then $V \setminus A_1 = \{y_i \mid 1 \leq i \leq n\}$ and

$cut(A_1, V\setminus A_1) = 1,$

$vol(A_1) = 2m-1,$



$$vol(V\setminus A_1) = n(n-1)+1$$

Then,

$$N_1 = Ncut(A_1, V\setminus A_1) = 1 \times \left(\frac{1}{2m-1} + \frac{1}{n(n-1)+1}\right)$$

$$= \frac{2m+n(n-1)}{[n(n-1)+1](2m-1)}$$

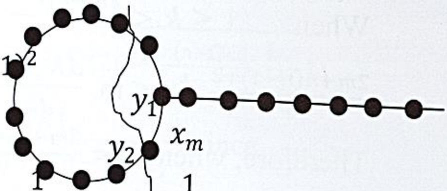
Case II:

Let $A_2 \subset V$ such that $A_2 = \{x_i \mid 1 \leq i \leq m\} \cup \{y_1\}$ then $V \setminus A_2 = \{y_i \mid 2 \leq i \leq n\}$ and

$cut(A_2, V\backslash A_2) = n - 1,$

$vol(A_2) = 2m + n - 1,$

$$vol(V\backslash A_2) = (n-1)^2$$



Then,

$$N_2 = Ncut(A_2, V\backslash A_2) = (n-1) \times \left(\frac{1}{2m+n-1} + \frac{1}{(n-1)^2}\right)$$

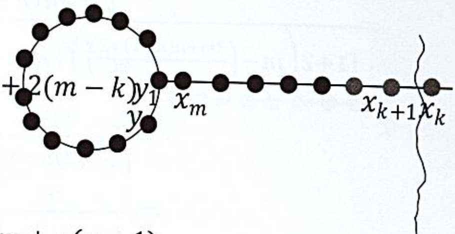
$$= \frac{2m + n(n-1)}{(n-1)(2m+n-1)}$$

Case III:

Let $A_3 \subset V$ such that $A_3 = \{x_i \mid 1 \leq i \leq k\}$ then $V \setminus A_3 = \{x_i \mid k+1 \leq i \leq m\} \cup \{y_i \mid 1 \leq i \leq n\}$ and $cut(A_3, V\backslash A_3) = 1,$

$vol(A_3) = 2k - 1,$

$$vol(V\backslash A_3) = n(n-1) + 1 + 2(m-k)$$



Then,

$$N_3(k) = Ncut(A_3, V\backslash A_3) = \frac{2m + n(n-1)}{[1 + 2(m-k) + n(n-1)](2k-1)}$$

For the minimum or maximum of N_3, differentiate $N_3(k)$ with respect to k.

$$\frac{dN_3(k)}{dk}$$

$$= \frac{-[2m + n(n-1)]\left[\left(1 + 2(m-k) + n(n-1)\right) \times 2 + (2k-1)(-2)\right]}{[1 + 2(m-k) + n(n-1)]^2 (2k-1)^2}$$

$$= \frac{-2[2m + n(n-1)][2m + n(n-1) + 2 - 4k]}{[1 + 2(m-k) + n(n-1)]^2 (2k-1)^2}$$

$\frac{dN_3(k)}{dk}$ =0 then

$k = \frac{2m+n(n-1)+2}{4}and1 \le k < m$

When $1 \le k < \frac{2m+n(n-1)+2}{4}$, $\frac{dN_3(k)}{dk} < 0$ and when $\frac{2m+n(n-1)+2}{4} \le k <$ m, $\frac{dN_3(k)}{dk} > 0$.

Therefore, when $k = \frac{2m+n(n-1)+2}{4}$, $N_3(k)$is the minimum value.

$$1 \le k < m$$

$$1 \le \frac{2m + n(n-1) + 2}{4} < m$$

$$m > \frac{n(n-1) + 2}{2}$$

Since k is an integer,

Minimum $N_3(k)$

$$= \frac{2m+n(n-1)}{[1+2\left(m-\left(\frac{2m+n(n-1)+2}{4}\right)\right)+n(n-1)](2(\frac{2m+n(n-1)+2}{4})-1)}$$

$$= \frac{[2m+n(n-1)]\times 4}{[2+(2m-n(n-1)-2)+2n(n-1)](2m+n(n-1)+2-2)}$$

$$= \frac{4}{(2m+n(n-1))} \text{since} m > \frac{n(n-1)+2}{2} \text{ and } k \in \mathbb{Z}$$

If k is not a integer,

Let $k' = k = \frac{2m+n(n-1)+2}{4} \pm \frac{1}{2}$ is integer

$\because 2m + n(n-1) + 2$ is an even number, therefore $\frac{2m+n(n-1)+2}{2}$should be integer.

$$N_3(k + \tfrac{1}{2}) = \frac{2m+n(n-1)}{[1+2\left(m-\left(\frac{2m+n(n-1)+2}{4}+\frac{1}{2}\right)\right)+n(n-1)](2(\frac{2m+n(n-1)+2}{4}+\frac{1}{2})-1)}$$

$$= \frac{[2m+n(n-1)]\times 4}{[2m+n(n-1)-2][2m+n(n-1)+2]} \quad \text{since } m > \frac{n(n-1)+2}{2} \text{ and } k \notin \mathbb{Z}$$

$$N_3(k-\tfrac{1}{2}) = \frac{2m+n(n-1)}{\left[1+2\left(m-\left(\frac{2m+n(n-1)+2}{4}-\frac{1}{2}\right)\right)+n(n-1)\right]\left(2\left(\frac{2m+n(n-1)+2}{4}-\frac{1}{2}\right)-1\right)}$$

$$= \frac{[2m+n(n-1)]\times 4}{[2m+n(n-1)+2][2m+n(n-1)-2]} \quad \text{since } m > \frac{n(n-1)+2}{2} \text{ and } k \notin \mathbb{Z}$$

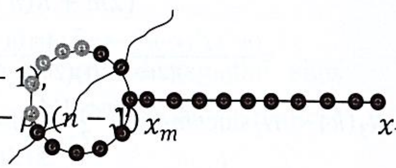
We have , $N_3\left(k+\frac{1}{2}\right) = N_3(k-\frac{1}{2})$

Case IV:

Let $A_4 \subset V$ such that $A_4 = \{y_i \mid j \le i \le j+\beta\}$ then $V \setminus A_4 = \{x_i \mid 1 \le i \le m\} \cup \{y_i \mid 1 \le i < j\} \cup \{y_i \mid j+\beta < i \le n\}$ and $cut(A_4, V\setminus A_4) = \beta(n-\beta)$,

$$vol(A_4) = \beta(n-1),$$

$$vol(V\setminus A_4) = 2m + (n-\beta)(n-1)$$



Then,

$$N_4 = Ncut(A_4, V\setminus A_4)$$

$$= \beta(n-\beta)\times\left(\frac{1}{\beta(n-1)} + \frac{1}{2m+(n-\beta)(n-1)}\right)$$

$$= \frac{(n-\beta)[2m+n(n-1)]}{[2m+(n-\beta)(n-1)](n-1)}$$

Comparing N_4 with N_2

$$N_4 - N_2 = \frac{(n-\beta)[2m+n(n-1)]}{[2m+(n-\beta)(n-1)](n-1)} - \frac{2m+n(n-1)}{(n-1)(2m+n-1)}$$

$$= \frac{[2m+n(n-1)]\{2m(n-\beta-1)\}}{[2m+(n-\beta)(n-1)](n-1)(2m+n-1)} \ge 0 \qquad \because \beta \le (n-1)$$

$$\therefore N_4 > N_2$$

Therefore we can ignore Case IV.
Comparing N_1 with N_2 and N_3,
By lemma 2(2)

$$N_1 \leq N_2$$

And lemma 2(1) implies, if $m \leq \frac{1}{2}(n^2 - n + 4)$ then $(\alpha - 1) > c(\alpha)$, $(2 \leq \alpha \leq m)$

$$\therefore N_1 < N_3$$

Therefore when $2 \leq m \leq \frac{1}{2}(n^2 - n + 4)$,

$$Mcut(LP_{n,m}) = N_1$$

If$m > \frac{1}{2}(n^2 - n + 4) > \frac{1}{2}(n^2 - n + 2)$,
By Lemma 2(1) we have,

$$N_3 < N_1$$

Compare the value of $N_3(k)$, $N_3\left(k + \frac{1}{2}\right)$ with N_2

$$N_3(k) - N_2 = \frac{4}{(2m + n(n-1))} - \frac{2m + n(n-1)}{(n-1)(2m + n - 1)}$$

$$= \frac{(n-1)^2(4-n^2) - 4m^2 + 4m(n-1)(2-n)}{(n-1)(2m+n-1)(2m+n(n-1))} < 0 \qquad \because n \geq 3$$

$N_3(k) < N_2$since$m > \frac{n(n-1)+4}{2}$ and $k \in \mathbb{Z}$

$$N_3\left(k + \frac{1}{2}\right) - N_2$$

$$= \frac{[2m + n(n-1)] \times 4}{[2m + n(n-1) - 2][2m + n(n-1) + 2]} - \frac{2m + n(n-1)}{(n-1)(2m + n - 1)}$$

$$= \frac{[2m + n(n-1)] \times [4(n-1)(2m + n - 1) - [2m + n(n-1) - 2][2m + n(n-1) + 2]}{[2m + n(n-1) - 2][2m + n(n-1) + 2](n-1)(2m + n - 1)}$$

$$= \frac{[2m + n(n-1)] \times [(n-1)^2(4 - n^2) + 4m(n-1)(2-n) - 4(m^2 - 1)]}{[(2m + n(n-1))^2 - 4](n-1)(2m + n - 1)}$$

$$< 0 \because n \geq 3, m \geq 1$$

$N_3\left(k + \frac{1}{2}\right) < N_2$since$m > \frac{n(n-1)+4}{2}$ and $k \notin \mathbb{Z}$

when $m > \frac{n(n-1)+4}{2}$,

$$\therefore Mcut(LP_{n,m}) = \begin{cases} N_3(k) & k \in \mathbb{Z} \\ N_3\left(k+\frac{1}{2}\right) & k \notin \mathbb{Z} \end{cases}$$

If $m = 1$

$N_1 = \frac{2+n(n-1)}{[n(n-1)+1]}$ dna $\quad N_2 = \frac{2+n(n-1)}{(n-1)(n+1)}$

($N_3(k)$ does not consider when $m = 1$, because we considered only k < m)

$$N_2 - N_1 = \frac{2+n(n-1)}{(n-1)(n+1)} - \frac{2+n(n-1)}{[n(n-1)+1]}$$

$$= \frac{[2+n(n-1)](2-n)}{(n-1)(n+1)[n(n-1)+1]} < 0$$

$N_2 < N_1$ when $m = 1$

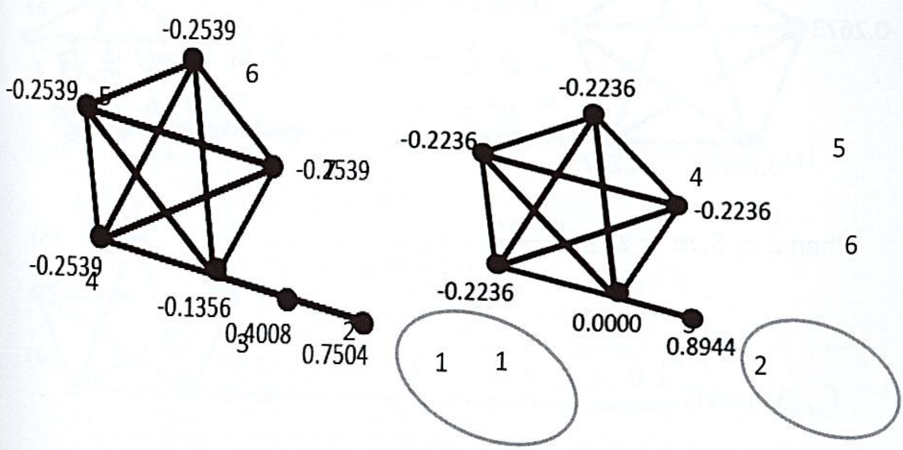
Therefore when = 1 ,

$$Mcut(LP_{n,m}) = \frac{2+n(n-1)}{(n-1)(n+1)}$$

sign graphs

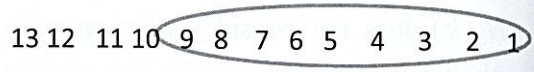
The graph $LP_{n,m}$ can be partitioned by considering the sign patterns of eigenvector corresponding to the second smallest eigenvalue of a difference Laplacian matrix.

(a).When$n = 5, m = 2$(b). When $n = 5, m = 1$



(c).$When\, n = 5, m = 13$

Figure 3:Second smallest eigenvector of differenceLaplacian matrix of $LP_{n,m}$



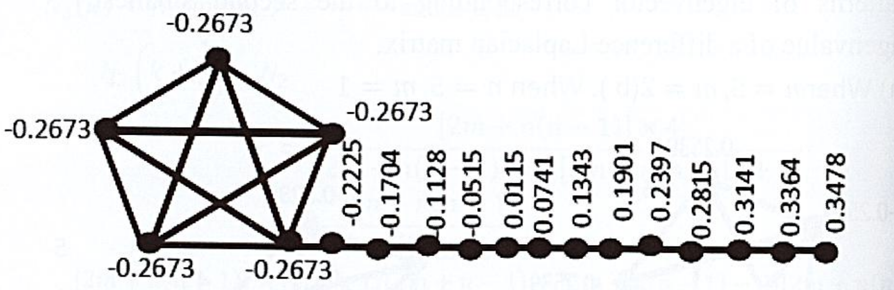
$Laplacian\, cut$:= Graph partition by considering the sign patterns of eigenvector correspond

to the second smallest eigenvalue of difference Laplacian matrix.

$Normalized\, cut$:= Graph partition by considering the sign patterns of eigenvector

correspond to thesecond smallest eigenvalue of normalized Laplacian

matrix.

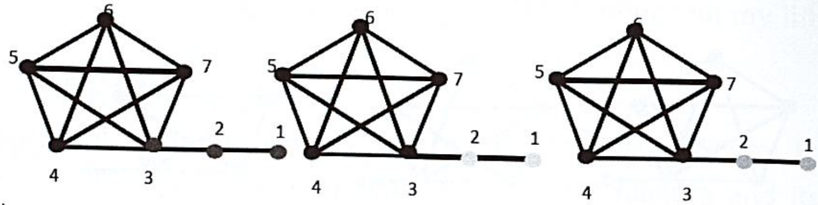


When $n = 5, m = 2 \leq \frac{n^2-n+4}{2}$

(a) (b)

(c)

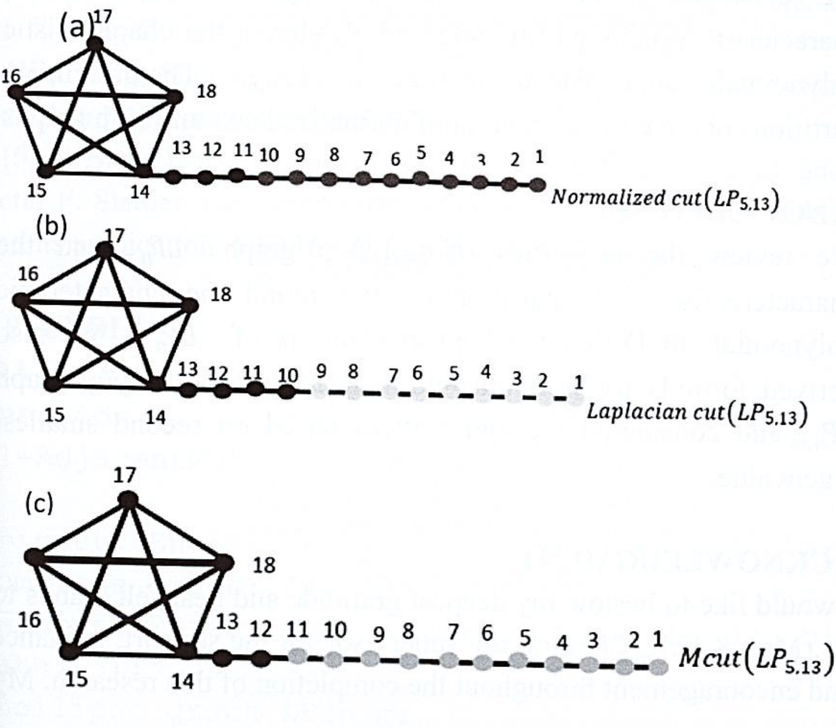
(b)



(b) (c)

$$Normalized\ cut(LP_{5,2}) Laplacian\ cut(LP_{5,2}) Mcut(LP_{5,2})$$

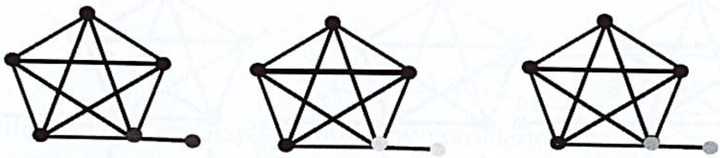
When $n = 5, m = 13 > \frac{n^2-n+4}{2}$



When $n = 5, m = 1$

(a) (b)

(c)



$Normalized\ cut(LP_{5,1}) Laplacian\ cut(LP_{5,1}) Mcut(LP_{5,1})$

Figure 4: $Normalized\ cut$, $Laplacian\ cut$ and $Mcut$ of some $LP_{n,m}$ graphs

DISCUSSION

We discussed properties of Lollipop graph $LP_{n,m}$ and the characteristics of its partitioning and alsoderive the characteristic polynomials of Laplacian matrix, of $LP_{n,m}$. Then compare partitions obtained from minimum normalized cut and sign graphs.

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We review the properties of Lollipop graph $LP_{n,m}$ and the characteristics of its partitioning. We found the characteristic polynomials of Difference Laplacian matrix of $LP_{n,m}$. We also derived formula for minimum normalized cut of Lollipop graph $LP_{n,m}$ and considered the sign pattern based on second smallest eigenvalue.

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APPENDIX

MATLAB coding for Adjacency matrix of $LP_{n,m}$

```
function [A n m]=AdjacentMatrix_LollipopGraph_2()

n=input('Enter the number of vertices of complete graph: ');
m=input('Enter the number of vertices of path: ');
%Lollipop graph LP(n,m)
```

```
%n=number of vertices of complete graph
%m=number of vertices of path
A=zeros(n+m);
for i=1:m
A(i,i+1)=1;
    A(i+1,i)=1;
end
 for j=m+1:m+n
for k=m+1:m+n
if k == j
A(k,k)=0;
else
            A(j,k)=1;
end
end
end
```

MATLAB coding for Laplacian matrix of $LP_{n,m}$

```
function
[D,L]=DifferenceLaplacian_LollipopGraph_2()

%without loop & multiple edges
% D-degree matrix
% L-Laplacian matrix
A=AdjacentMatrix_LollipopGraph_2();
%W=weightAdjacentMatrix_LollipopGraph();
p=size(A);
d=sum(A');
D=zeros(p);
L=zeros(p);
for i=1:p
```

```
D(i,i)=d(i);
L(i,i)=d(i);
end
for i=1:p
for j=1:p
if A(i,j)==1
     L(i,j)=-1;
end
end
end
```

MATLAB coding for Normalized Laplacian matrix of $LP_{n,m}$

```
%normalized laplacian matrix of lollipop graph

Function [NL D]=NormalizedLaplacian_LollipopGraph ()

%without loop & multiple edges
% D-degree matrix
% L-Laplacian matrix
% NL-Normalized Laplacian matrix

A=AdjacentMatrix_LollipopGraph_2();
[D,L]=DifferenceLaplacian_LollipopGraph_2();
n=size(A);
NL=zeros(n);
NL=D^(-1/2)*(D-A)*D^(-1/2);

eigenvalues=eig(NL);
```